

Multi-dimensional scalar balance laws with discontinuous flux

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Abstract

We consider the problem of existence of entropy weak solutions to scalar balance laws with a dissipative source term. The flux function may be discontinuous with respect both to the space variable x and the unknown quantity u . The problem is formulated in the framework of multi-valued mappings. We use the notion of entropy-measure valued solutions to prove the so-called contraction principle and comparison principle.

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1 Introduction

Our interest is directed to the following Cauchy problem describing the evolution of $u : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}$

$$u_t + \operatorname{div} \Phi(x, u) \ni f(t, x, u) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^N, \quad (1.1)$$

$$u(0, \cdot) = u_0 \quad \text{on } \mathbb{R}^N. \quad (1.2)$$

where $\Phi : \mathbb{R}^N \times \mathbb{R} \rightarrow 2^{\mathbb{R}^N}$ is a multi-valued mapping and $f : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a source term. Moreover $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given initial data. The assumptions for Φ and f shall be presented below. The formulation of the problem in the language of multi-valued flux function allows to capture relations which are not necessarily functions.

We will assume that the flux function is in the form of a composition, which allows, with an appropriate change of variables, to formulate the definition of entropy weak solutions in terms of the new variables. An important property of such defined solutions is that in case of smooth fluxes they correspond to the classical definition of entropy weak solutions, see e.g. Kružkov [15]. We assume about Φ and f that:

(H1) $\Phi(x, u)$ is a multi-valued mapping given by the formula $\Phi(x, u) = A(\theta(x, u))$ where $A : \mathbb{R} \rightarrow \mathbb{R}^N$, A is continuous and $\theta : \mathbb{R}^N \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \emptyset$ is a multi-valued mapping such that, for almost every $x \in \mathbb{R}^N$, $\theta(x, \cdot) : \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \emptyset$ is a maximal monotone operator with $0 \in \theta(x, 0)$. The inverse to θ (w.r.t u), which we call η , is continuous. Moreover, we assume that

$$\theta^*(\cdot, l) \in L^1(\mathbb{R}^N) \quad (1.3)$$

for each $l \in \mathbb{R}$, where θ^* denotes the minimal selection of the graph of θ .

(H2) there exist continuous functions h_1 and h_2 with $\lim_{|u| \rightarrow \infty} h_1(u) = \infty$ such that

$$h_1(u) \leq |\bar{\theta}| \leq h_2(u) \quad (1.4)$$

for all $\bar{\theta} \in \theta(x, u)$, almost every $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$

(H3) there exists $1 \leq p \leq \frac{N}{N-1}$ and constants $R_\infty > 0$ and $C_\infty > 0$ such that for all $x > R_\infty$

$$|A(s)|^p \leq C_\infty |\eta(x, s)|$$

(H4) $f(\cdot, \cdot, u) \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^N)$ for all $u \in \mathbb{R}$; $f(t, x, \cdot)$ is continuous and $f(t, x, 0) = 0$ for a.a. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Moreover f is dissipative ($-f$ is monotone w.r.t. the last variable), i.e.,

$$(f(t, x, u) - f(t, x, v))(u - v) \leq 0 \quad \text{for all } u, v \in \mathbb{R} \text{ and a.a. } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \quad (1.5)$$

Remark 1.1 *One could consider a more general source term, namely for almost all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$ a maximal monotone (possibly multi-valued) mapping f . Then we would rewrite (1.1) as $u_t + \operatorname{div} \Phi(x, u) - f(t, x, u) \ni 0$. The scalar conservation laws with a multi-valued source term were considered e.g. in [12].*

The approach of considering the flux function in form of a composition was used by Panov in [17] to solve the problem of well-posedness for a scalar conservation law without source term (i.e. $f = 0$) and a flux function discontinuous with respect to x . More precisely, the author assumed that $\Phi(x, u) = A(\theta(x, u))$, where $A \in \mathcal{C}(\mathbb{R}; \mathbb{R}^N)$ and $\theta : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, which is for almost all $x \in \mathbb{R}^N$ strictly increasing with respect to u . Moreover the same condition as (H2) was assumed. Hence if $\eta(x, v)$ is the inverse to θ , i.e., $\theta(x, \eta(x, v)) = v$ then u is a solution to (1.1)–(1.2) with $f = 0$ if there exists v such that $u = \eta(x, v)$ and the following entropy inequality is satisfied in the distributional sense in $\mathbb{R}_+ \times \mathbb{R}^N$ for all $k \in \mathbb{R}$

$$|\eta(x, v) - \eta(x, k)|_t + \operatorname{div} (\operatorname{sgn} (v - k)(A(v) - A(k))) \leq 0. \quad (1.6)$$

The corresponding approach we find for fluxes discontinuous only with respect to u in the paper by Carrillo, [7]. The author studied the problem in a bounded domain

$$\begin{aligned} u_t + \operatorname{div} \Phi(u) &\ni f && \text{in } (0, T) \times \Omega \\ u(0) &= u_0 && \text{in } \Omega \end{aligned} \quad (1.7)$$

under the assumption that Φ is allowed to have discontinuities of first type on a finite subset of \mathbb{R} . After a change of variables the author deals with the following problem

$$\begin{aligned} g(v)_t + \operatorname{div} \Psi(u) &= f \quad \text{in } (0, T) \times \Omega, \\ g(v(0)) &= u_0 \quad \text{in } \Omega. \end{aligned} \tag{1.8}$$

The proof of existence of solutions bases upon the comparison principle and the entropy inequality involving a version of semi Kruřkov entropies, namely $E(v, k) = (g(v) - g(k))^+$.

The similar problem was considered in Bulíček et al. [6] with the use of different approach, namely

$$\begin{aligned} u_t + \operatorname{div} \Phi(u) &= 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0) &= u_0 \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{1.9}$$

The authors showed existence and uniqueness of entropy weak solutions for jump continuous Φ (i.e. having countable, not necessarily finite, number of jumps). For the proof they essentially used the method of entropy measure-valued solutions introduced by DiPerna, cf. [9] and later extended by Szepessy in [20]. To handle the discontinuity of the flux function Bulíček et al. showed existence of a parametrization U , namely a nondecreasing function such that $\Phi \circ U$ is continuous.

These ideas are combined in [5], where the authors treat the case of a flux function discontinuous in x and u for the problem

$$\begin{aligned} u_t + \operatorname{div} \Phi(x, u) &= 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0) &= u_0 \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{1.10}$$

The set of assumptions corresponds to the one formulated by Panov in [17], namely $\Phi(x, u) = A(\theta(x, u))$ extended by the possibility that A is a jump continuous function. Again through appropriate estimates for entropy measure-valued solutions and finding the parametrization U the authors showed well-posedness for (1.10). Both in [5] and [6] the uniqueness of entropy weak solutions needs to be understood up to the level sets of the parametrization U . This is also related with a restricted family of entropies which are allowed, what we will discuss in more detail after the statement of definition and main theorem.

In the present paper we have added a source term, which requires additional attention in various crucial estimates. However the main novelty is to combine the approaches from [5] and [7] and consequently obtain a stronger result. The proof bases on the combination of comparison principle and formulating the definition with help of the entropies of semi-Kruřkov type with compactness arguments. The approach presented here gives additional advantages. If the starting point are considerations on the problem formulated with discontinuous flux (jump continuous), we shall first fill up the jumps. In the case of [5] we may only do it with intervals, however in the current setting we have more freedom. We come back to this issue at the end of the introduction, after formulating the definition and recalling in more detail the framework of [5].

Before we formulate the definition of entropy weak solutions let us introduce some notation. By $\mathcal{D}(\Omega)$ we mean the set of smooth functions with a compact support in Ω , $\mathcal{C}(\Omega; X)$ is the set of continuous functions from Ω to the space X . For $1 \leq p \leq \infty$ by $L^p(\Omega)$ we understand standard Lebesgue spaces and by $L^p(\mathbb{R}_+; X)$ Bochner spaces.

Definition 1.1 *Let Φ, f satisfy the assumptions (H1)–(H4). We say that a function $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N) \cap L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^N))$ is an entropy weak solution of (1.1)–(1.2) if there exists a function $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$ such that $u = \eta(x, g)$ and for all $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^N)$, $\psi \geq 0$ and for all $k \in \mathbb{R}$*

(i)

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathbb{R}^N} \{(\eta(x, g) - \eta(x, k))^+ \psi_t + \chi_{\{g > k\}}(A(g) - A(k)) \nabla \psi + \chi_{\{g > k\}} f \psi\} \\ \geq - \int_{\Omega} (u_0 - \eta(x, k))^+ \psi(0, \cdot), \end{aligned} \quad (1.11)$$

(ii)

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathbb{R}^N} \{(\eta(x, k) - \eta(x, g))^+ \psi_t + \chi_{\{k > g\}}(A(k) - A(g)) \nabla \psi - \chi_{\{k > g\}} f \psi\} \\ \geq - \int_{\mathbb{R}^N} (\eta(x, k) - u_0)^+ \psi(0, \cdot). \end{aligned} \quad (1.12)$$

Remark 1.2 *Note that (i) and (ii.) of Definition 1.1 are equivalent to the conditions*

$$\int_{\mathbb{R}_+ \times \mathbb{R}^N} |\eta(x, g) - \eta(x, k)| \psi_t + \operatorname{sgn}(g - k)(A(g) - A(k)) \nabla \psi + \operatorname{sgn}(g - k) f \psi \geq 0 \quad (1.13)$$

for all $\psi \in \mathcal{D}((0, T) \times \mathbb{R}^N)$ such that $\psi \geq 0$ and

$$\operatorname{ess} \lim_{t \rightarrow 0} \int_K |u(t, x) - u_0| \, dx = 0 \quad (1.14)$$

for any compact $K \subset \mathbb{R}^N$.

Now we are ready to formulate the main result of the paper on the existence of entropy weak solutions.

Theorem 1.3 *Let Φ, f satisfy the assumptions (H1)–(H4). Assume $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there exists an entropy weak solution u to (1.1)–(1.2) in the sense of Definition 1.1*

To understand the advantage of the framework presented here let us closely observe the approach in [5] and recall that by an *admissible parametrization* of A the authors understand a couple (\mathcal{A}, U) if the function $U \in \mathcal{C}(\mathbb{R})$ is nondecreasing and $\lim_{s \rightarrow \pm\infty} U(s) = \pm\infty$. Moreover, defining

$$\alpha_k := \inf_{\alpha; U(\alpha)=z_k} \alpha, \quad \beta_k := \sup_{\beta; U(\beta)=z_k} \beta \quad (1.15)$$

it is required that the function U is constant on $[\alpha_k, \beta_k]$ and strictly increasing on (β_k, α_{k+1}) for all $k \in \mathbb{N}$. The function $\mathcal{A} \in \mathcal{C}(\mathbb{R}; \mathbb{R}^N)$ satisfies $\mathcal{A}(s) \in A(U(s))$ and is linear on $[\alpha_k, \beta_k]$ for all $k \in \mathbb{N}$. Then $u \in L^\infty(0, \infty; L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$ is an entropy weak solution to (1.1) related to (A, θ) and u_0 for an admissible parametrization (\mathcal{A}, U) of A if there exists a function $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$ such that

$$\eta(x, U(g(t, x))) = u(t, x), \quad \mathcal{A}(g(t, x)) \in A(\theta(x, u(t, x))) \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}^N, \quad (1.16)$$

$$\text{ess lim}_{t \rightarrow 0} \int_K |u(t, x) - u_0(x)| dx = 0, \quad \text{for any compact } K \subset \mathbb{R}^N, \quad (1.17)$$

and for all nonnegative $\psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^N)$ and arbitrary $k \in \mathbb{R} \setminus \bigcup_{l \in \mathbb{N}} (\alpha_l, \beta_l)$ there holds

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} |\eta(x, U(g(t, x))) - \eta(x, U(k))| \psi_t(t, x) dx dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\text{sgn}(g(t, x) - k)(\mathcal{A}(g(t, x)) - \mathcal{A}(k))) \cdot \nabla \psi(t, x) dx dt \geq 0. \end{aligned} \quad (1.18)$$

The numbers $\alpha_l, \beta_l, l \in \mathbb{N}$ are defined in (1.15).

Remark 1.4 (Remark 1.1 from [5]) Any entropy weak solution is a weak solution to (1.1)-(1.2). Indeed, since $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$ we may take $k := \pm \|g\|_\infty$ in (1.18) (or possibly we increase/decrease the value of k such that U is strictly increasing in k) and by using the strict monotonicity of η and the monotonicity of U we conclude that

$$u_t + \text{div } A(g) = 0, \quad \text{in the sense of distribution in } \mathbb{R}_+ \times \mathbb{R}^N, \quad (1.19)$$

which is exactly (1.1) with $f = 0$. Next, we can use the fact that by functions $|u - \cdot|$ one can generate any convex function and therefore it is a direct consequence of (1.18) that (see [6] for details) for all smooth convex E , such that E is linear on (α_k, β_k) for all \mathbb{N} , where α_k and β_k are introduced in (1.15), there holds

$$Q_u(x, g)_t + \text{div } Q_A \leq 0, \quad \text{in sense of distribution in } \mathbb{R}_+ \times \mathbb{R}^N \quad (1.20)$$

with Q_u and Q_A given by

$$\partial_s Q_u(x, s) = \partial_s \eta(x, U(s)) E'(s), \quad Q_A(s) = \int_0^s A'(\tau) E'(\tau) d\tau. \quad (1.21)$$

¹Since A is only continuous, then this relation should be understood as follows $Q_A(s) = A(s)E'(s) - \int_0^s A(\tau)E''(\tau) d\tau$.

Hence from here one easily observes that (1.18) does not hold for all $k \in \mathbb{R}$ and the family of entropies is restricted to such that are linear on the intervals (α_k, β_k) . In a consequence we lose the information on the intervals where θ is multi-valued. In the current paper the situation is significantly different. The approximation of the problem follows in two steps. One is the mollification of the multi-valued term (we take a minimal selection and then mollify with a smooth kernel) and the second one consists in subtracting a strictly monotone perturbation from the source term. Then the right hand side becomes strictly dissipative, namely the inequality in (1.5) becomes strict for $u \neq v$ and this is the sufficient argument to obtain the uniqueness of entropy measure-valued solutions and to show they reduce to a Dirac measure. Here one needs the initial condition. For passing to the limit with a perturbation of the right-hand side one takes advantage of the semi-Kružkov entropies $E(u, k) = (u - k)^+$ and $E(u, k) = (u - k)^-$ and then combines the information on the monotonicity of appropriate sequences and boundedness to obtain the strong convergence. Hence this is sufficiently powerful information to provide that on the sets where θ is multi-valued one is not obliged to have linear (or affine) functionals and continuity is enough for the limit passage.

We complete this section by referring to other previous results for scalar conservation laws with discontinuous fluxes. The approach of Panov [17] arises from an idea of *adapted entropies* introduced for the problems with x -discontinuous fluxes in [4] and later in [3]. The approach consisted in using in classical Kružkov entropies in place of a constant k the solution to a stationary problem. The equivalence between such solutions and entropy weak solutions understood as in [15] in case of smooth fluxes was shown in [8]. There are various different approaches to fluxes discontinuous in x , see e.g. the front tracking method for one dimensional problem, cf. [11, 14, 19]. The multi-dimensional problem was considered among others in [2, 13, 16]. To motivate the studies in the direction of fluxes discontinuous in u we refer to the implicit constitutive theory and the works of Rajagopal, [18], described also in more detail in [6].

The paper is organized as follows. In Section 2 we collect all the essential tools needed for the proof of Theorem 1.3. We start with a contraction principle formulated for entropy measure-valued solutions (Lemma 2.1). Then essentially using this result we show a contraction principle for entropy weak solutions (Lemma 2.2, estimate (2.45)) and comparison principle for entropy weak solutions (Lemma 2.2, estimate (2.46)). The whole Section 3 is dedicated to the proof of Theorem 1.3. We start with regularizing the flux function and then add the strictly monotone perturbation to the source term. The scheme of the proof is first showing the existence of entropy measure-valued solutions, then their uniqueness and finally concluding that the solutions are indeed entropy weak solutions. In the final part of the paper there is an appendix which partially recalls the facts from [5] and also extends some technical lemmas for the case of multi-valued mappings.

2 Entropy inequalities

We shall start this section with the definition of entropy measure-valued solutions and then collect the essential estimates used for the proof of existence of solutions: averaged contraction principle and comparison principle.

2.1 Averaged contraction principle for entropy measure valued solutions

We recall that $\mathcal{M}(\mathbb{R})$ denotes the space of bounded Radon measures and $\text{Prob}(\mathbb{R})$ the space of probability measures, $\mathcal{C}_b(\mathbb{R})$ stands for the space of continuous bounded functions. As usual, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{C}_b(\mathbb{R})$ and $\mathcal{M}(\mathbb{R})$. By a Young measure ν we mean a weak* measurable map $\nu : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathcal{M}(\mathbb{R})$ and such that $\nu_{(t,x)} \geq 0$, $\|\nu_{(t,x)}\|_{\mathcal{M}(\mathbb{R})} \leq 1$ for a.a. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Any bounded sequence of measurable functions $u^n : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}$ generates a Young measure, which is a probability measure. By $L_w^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \mathcal{M}(\mathbb{R}))$ we understand the space of weak* measurable maps $\nu : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathcal{M}(\mathbb{R})$ that are essentially bounded.

Definition 2.1 *Let Φ, f satisfy the assumptions (H1)–(H4) and $u_0 \in L_{loc}^1(\mathbb{R}^N)$. We say that a Young measure $\nu : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \text{Prob}(\mathbb{R})$ is an entropy measure-valued solution to (1.1) if there exists $R(t, x) \in L_{loc}^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$ such that*

$$\text{supp } \nu_{(t,x)} \subset [-R(t, x), R(t, x)] \quad \text{for a.a. } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \quad (2.22)$$

and if for all $\mu \in \mathbb{R}$ and all nonnegative $\psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^N)$ there holds

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle (\eta(x, \lambda) - \eta(x, \mu))^+, \nu_{(t,x)}(\lambda) \rangle \psi_t(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda > \mu\}} (A(\lambda) - A(\mu)), \nu_{(t,x)}(\lambda) \rangle \cdot \nabla \psi(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda > \mu\}} f(t, x, \lambda), \nu_{(t,x)}(\lambda) \rangle \psi \, dx \, dt \geq 0. \end{aligned} \quad (2.23)$$

and

$$\begin{aligned} & - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle (\eta(x, \lambda) - \eta(x, \mu))^- , \nu_{(t,x)}(\lambda) \rangle \psi_t(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda < \mu\}} (A(\lambda) - A(\mu)), \nu_{(t,x)}(\lambda) \rangle \cdot \nabla \psi(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda < \mu\}} f(t, x, \lambda), \nu_{(t,x)}(\lambda) \rangle \psi \, dx \, dt \leq 0. \end{aligned} \quad (2.24)$$

Moreover, for all compact $K \subset \mathbb{R}^N$ the following holds

$$\text{ess } \lim_{t \rightarrow 0^+} \int_K \langle |\eta(x, \lambda) - u_0(x)|, \nu_{(t,x)}(\lambda) \rangle \, dx = 0. \quad (2.25)$$

The existence of entropy measure-valued solutions will be a byproduct of the proof of existence of entropy weak solutions. Below we formulate and prove the estimate (the averaged contraction principle) which is used both for showing existence and uniqueness of entropy measure-valued solutions. The proof bases on the method of doubling the variables, but on the level of measure-valued solutions.

Lemma 2.1 *Assume that ν, σ are two local entropy measure-valued solutions to (1.1) with a right-hand side f and initial condition $u_0 \in L^1_{loc}(\mathbb{R}^N)$. Let² $E(\xi) = |\xi|$ with a corresponding flux $Q(\lambda, \mu) = \text{sgn}(\lambda - \mu)(A(\lambda) - A(\mu))$. Moreover let*

$$E'(\xi) := (\partial E)^0(\xi) = \begin{cases} -1 & \text{for } \xi < 0 \\ 0 & \text{for } \xi = 0 \\ 1 & \text{for } \xi > 0 \end{cases}.$$

Then for all nonnegative $\psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^N)$ it holds

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \psi_t(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle Q(\lambda, \mu), \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \cdot \nabla \psi(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E'(\lambda - \mu)(f(t, x, \lambda) - f(t, x, \mu)), \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \psi(t, x) \, dx \, dt \geq 0 \end{aligned} \quad (2.26)$$

Proof: Let $\omega \in \mathcal{D}(-1, 1)$ be a regularizing kernel, i.e., $\omega(x) = \omega(-x)$ and $\int_{-1}^1 \omega(x) \, dx = 1$. Then, for any $\gamma > 0$, we define

$$\begin{aligned} \omega_1^\gamma(t) &:= \gamma^{-1} \omega(t/\gamma) && \text{for all } t \in \mathbb{R}, \\ \omega_2^\gamma(x) &:= \gamma^{-N} \omega(x_1/\gamma) \cdot \dots \cdot \omega(x_N/\gamma) && \text{for all } x = (x_1, \dots, x_N) \in \mathbb{R}^N. \end{aligned}$$

For arbitrary $\varepsilon, \delta > 0$ we set $\omega^{\delta, \varepsilon}(t, x) := \omega_1^\delta(t) \cdot \omega_2^\varepsilon(x)$. Notice that for any Young measure $\nu \in L_w^\infty([0, T] \times \mathbb{R}^N; \mathcal{M}(\mathbb{R}))$ there exists a Young measure $\nu^\delta \in L_w^\infty(\mathbb{R}^N; \mathcal{C}^\infty([0, T]; \mathcal{M}(\mathbb{R})))$ with $\|\nu^\delta\|_{L_w^\infty([0, T] \times \mathbb{R}^N; \mathcal{M}(\mathbb{R}))} \leq 1$ such that for any $f \in \mathcal{C}_b(\mathbb{R})$ the following holds³ ($\omega_1^\delta * \langle f, \nu \rangle = \langle f, \nu^\delta \rangle$ for almost all $t \in \mathbb{R}$). Moreover, we can interchange the derivative as $\langle f, \partial_t \nu^\delta \rangle = \langle f, \nu^\delta \rangle_t$ for all $t \in \mathbb{R}$. Similarly, there exists $\nu^\varepsilon \in L_w^\infty([0, T]; \mathcal{C}^\infty(\mathbb{R}_{loc}^N; \mathcal{M}(\mathbb{R})))$ with $\|\nu^\varepsilon\|_{L_w^\infty([0, T] \times \mathbb{R}^N; \mathcal{M}(\mathbb{R}))} \leq 1$ such that $\omega_2^\varepsilon * \langle f, \nu \rangle = \langle f, \nu^\varepsilon \rangle$ and $\langle f, \partial_{x_i} \nu^\varepsilon \rangle = \partial_{x_i} \langle f, \nu^\varepsilon \rangle$ for all $x \in \mathbb{R}^N$, see Ref. [9].

Let $Q(\lambda, \mu) := \text{sgn}(\lambda - \mu)(A(\lambda) - A(\mu))$. Then

$$(\nu, \sigma) \mapsto \langle Q(\lambda, \mu), \nu \otimes \sigma \rangle \in \mathbb{R}$$

²We keep the general notation (E, Q) instead of writing the concrete form of the entropy and the entropy flux for the sake of the next lemmas and their proofs, where similar arguments are partially repeated.

³We extend the measure for $t < 0$ and $t > T$ by zero.

is a bounded bilinear form from $\mathcal{M}(\mathbb{R}) \times \mathcal{M}(\mathbb{R})$ to \mathbb{R} and

$$(t, x) \mapsto \nu_{(t,x)}^{\varepsilon, \delta} \in \mathcal{C}^\infty(K, (\mathcal{M}(\mathbb{R}), \|\cdot\|_{\mathcal{M}}))$$

$$(t, x) \mapsto \sigma_{(t,x)}^{\varepsilon, \delta} \in \mathcal{C}^\infty(K, (\mathcal{M}(\mathbb{R}), \|\cdot\|_{\mathcal{M}}))$$

for any compact $K \subset \mathbb{R}_+ \times \mathbb{R}^N$ and then

$$\operatorname{div} \langle Q(\lambda, \mu), \nu_{(t,x)}^\varepsilon \otimes \sigma_{(t,x)}^\varepsilon \rangle = \langle Q(\lambda, \mu), \nabla \nu_{(t,x)}^\varepsilon \otimes \sigma_{(t,x)}^\varepsilon \rangle + \langle Q(\lambda, \mu), \nu_{(t,x)}^\varepsilon \otimes \nabla \sigma_{(t,x)}^\varepsilon \rangle. \quad (2.27)$$

For arbitrary nonnegative $\psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^N)$ we observe that for all $\mu \in \mathbb{R}$

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}(\lambda) \rangle (\psi * (\omega_1^\delta \cdot \omega_2^\varepsilon))_t \, dx \, dt \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}^N} \omega_2^\varepsilon * \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}^\delta(\lambda) \rangle \psi_t \, dx \, dt. \end{aligned} \quad (2.28)$$

Similarly, we obtain for all $\mu \in \mathbb{R}$

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle Q(\lambda, \mu), \nu_{(t,x)}(\lambda) \rangle \cdot \nabla (\psi * (\omega_1^\delta \cdot \omega_2^\varepsilon)) \, dx \, dt = \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle Q(\lambda, \mu), \nu_{(t,x)}^{\delta, \varepsilon}(\lambda) \rangle \cdot \nabla \psi \, dx \, dt. \end{aligned} \quad (2.29)$$

Moreover

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E'(\lambda - \mu) f(t, x, \lambda), \nu_{(t,x)}(\lambda) \rangle \psi * (\omega_1^\delta \cdot \omega_2^\varepsilon) \, dx \, dt \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\lambda - \mu) f(t, x, \lambda), \nu_{(t,x)}(\lambda) \rangle \psi \, dx \, dt. \end{aligned} \quad (2.30)$$

Summing (2.23) and (2.24) we obtain an entropy inequality with the entropy $E(\xi) = |\xi|$, where we may take $\psi * (\omega_1^\delta \cdot \omega_2^\varepsilon)$ as a test function and using (2.28)–(2.30) we deduce that for all $\mu \in \mathbb{R}$ and all nonnegative $\psi \in \mathcal{D}((\varepsilon, \infty) \times \mathbb{R}^N)$ there holds

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} \omega_2^\varepsilon * \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}^\delta(\lambda) \rangle \psi_t \, dx \, dt \\ &+ \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle Q(\lambda, \mu), \nu_{(t,x)}^{\delta, \varepsilon}(\lambda) \rangle \cdot \nabla \psi \, dx \, dt \\ &+ \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\lambda - \mu) f(t, x, \lambda), \nu_{(t,x)}(\lambda) \rangle \psi \, dx \, dt \geq 0 \end{aligned} \quad (2.31)$$

which in particular implies that for all $\tilde{\mu} \in \mathbb{R}$ and all $(t, x) \in (\varepsilon, \infty) \times \mathbb{R}^N$ there holds

$$\begin{aligned} & (\omega_2^\varepsilon * \langle E(\eta(x, \lambda) - \eta(x, \tilde{\mu})), \nu_{(t,x)}^\delta(\lambda) \rangle)_t + \operatorname{div} \langle Q(\lambda, \tilde{\mu}), \nu_{(t,x)}^{\delta, \varepsilon}(\lambda) \rangle \\ & \leq (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\lambda - \tilde{\mu}) f(t, x, \lambda), \nu_{(t,x)}(\lambda) \rangle. \end{aligned} \quad (2.32)$$

Similarly we have for any $\varepsilon > 0$, $\tilde{\lambda} \in \mathbb{R}$ and all $(t, x) \in (\varepsilon, \infty) \times \mathbb{R}^N$

$$\begin{aligned} & \left(\omega_2^\varepsilon * \langle E(\eta(x, \tilde{\lambda}) - \eta(x, \mu)), \sigma_{(t,x)}^\delta(\mu) \rangle \right)_t + \operatorname{div} \langle Q(\tilde{\lambda}, \mu), \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \\ & \leq (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\mu - \tilde{\lambda})f(t, x, \mu), \sigma_{(t,x)}(\mu) \rangle. \end{aligned} \quad (2.33)$$

We apply $\sigma_{(t,x)}^{\delta,\varepsilon}$ to (2.32). Note that the left-hand side is a continuous function of μ and the right-hand side is only a Borel function of μ . Similarly we apply $\nu_{(t,x)}^{\delta,\varepsilon}$ onto (2.33). Summing the resulting expressions we find that for all $(t, x) \in (2\varepsilon, \infty) \times \mathbb{R}^N$ there holds

$$\begin{aligned} & \langle \omega_2^\varepsilon * \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}^\delta(\lambda) \rangle_t, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \\ & + \langle \omega_2^\varepsilon * \langle E(\eta(x, \lambda) - \eta(x, \mu)), \sigma_{(t,x)}^\delta(\mu) \rangle_t, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle \\ & + \operatorname{div} \langle Q(\lambda, \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \otimes \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \\ & \leq \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\lambda - \mu)f(t, x, \lambda), \nu_{(t,x)}(\lambda) \rangle, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \\ & + \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\mu - \lambda)f(t, x, \mu), \sigma_{(t,x)}(\mu) \rangle, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle =: I \end{aligned} \quad (2.34)$$

To proceed with a right-hand side we define the errors as follows

$$\begin{aligned} \mathcal{R}_{\varepsilon,\delta,n}^\lambda & := \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\lambda - \mu)(f(t, x, \lambda) - f^n(t, x, \lambda)), \nu_{(t,x)}(\lambda) \rangle, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \\ & + \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\lambda - \mu)f^n(t, x, \lambda), \nu_{(t,x)}(\lambda) \rangle, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \\ & - \langle \langle E'(\lambda - \mu)f^n(t, x, \lambda), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \end{aligned} \quad (2.35)$$

and

$$\begin{aligned} \mathcal{R}_{\varepsilon,\delta,n}^\mu & := \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\mu - \lambda)(f(t, x, \mu) - f^n(t, x, \mu)), \sigma_{(t,x)}(\mu) \rangle, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle \\ & + \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\mu - \lambda)f^n(t, x, \mu), \sigma_{(t,x)}(\mu) \rangle, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle \\ & - \langle \langle E'(\mu - \lambda)f^n(t, x, \mu), \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle \end{aligned} \quad (2.36)$$

where $(f^n)_{n \in \mathbb{N}}$ is the sequence of uniformly continuous functions in (t, x) and continuous in u and there exists an $L_K(n)$ such that for a fixed compact K it vanishes as $n \rightarrow \infty$ and

$$\sup_{\lambda \in K} \|f(\cdot, \cdot, \lambda) - f^n(\cdot, \cdot, \lambda)\|_{L^1(\mathbb{R}_+ \times \mathbb{R}^N)} \leq L_K(n). \quad (2.37)$$

Let \mathcal{W}_K^n be a modulus of continuity of the function f^n , namely $\mathcal{W}_K^n : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is continuous, $\mathcal{W}_K^n(0, 0) = 0$ and

$$\sup_{\lambda \in K} |f^n(t - s, x - y, \lambda) - f^n(t, x, \lambda)| \leq \mathcal{W}_K^n(|s|, |y|) \quad (2.38)$$

where K is an arbitrary compact subset of \mathbb{R} . Hence

$$\begin{aligned} I & = \langle \langle E'(\lambda - \mu)f^n(t, x, \lambda), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle + \mathcal{R}_{\varepsilon,\delta,n}^\lambda \\ & + \langle \langle E'(\mu - \lambda)f^n(t, x, \mu), \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle + \mathcal{R}_{\varepsilon,\delta,n}^\mu \end{aligned} \quad (2.39)$$

and as $E'(\xi) = -E'(-\xi)$ and using the Fubini theorem we further conclude

$$I = \langle \langle E'(\lambda - \mu)(f^n(t, x, \lambda) - f^n(t, x, \mu)), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle + \mathcal{R}_{\varepsilon,\delta,n}^\lambda + \mathcal{R}_{\varepsilon,\delta,n}^\mu. \quad (2.40)$$

Note that the function $E'(\lambda - \mu)(f^n(t, x, \lambda) - f^n(t, x, \mu))$ is continuous, although $E'(\lambda - \mu) = \text{sgn}(\lambda - \mu)$ is not continuous for $\lambda - \mu = 0$. We shall estimate the error $\mathcal{R}_{\varepsilon,\delta,n}^\lambda$, the estimates for $\mathcal{R}_{\varepsilon,\delta,n}^\mu$ follow the same lines. Then

$$\begin{aligned} |\mathcal{R}_{\varepsilon,\delta,n}^\lambda| &\leq | \langle \omega_1^\delta \cdot \omega_2^\varepsilon * \langle E'(\lambda - \mu)(f(t, x, \lambda) - f^n(t, x, \lambda)), \nu_{(t,x)}(\lambda) \rangle, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle | \\ &\quad + | \int_{\mathbb{R}} \int_{\mathbb{R} \times \mathbb{R}^N} \omega_1^\delta(s) \omega_2^\varepsilon(y) \langle E'(\lambda - \mu) f^n(t - s, x - y, \lambda), \nu_{(t-s,x-y)}(\lambda) \rangle dy ds d\sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \\ &\quad - \int_{\mathbb{R}} \langle E'(\lambda - \mu) f^n(t, x, \lambda), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \rangle d\sigma_{(t,x)}^{\delta,\varepsilon}(\mu) | \\ &\leq \sup_{\lambda \in K} \|f(\cdot, \cdot, \lambda) - f^n(\cdot, \cdot, \lambda)\|_{L^1(\mathbb{R}_+ \times \mathbb{R}^N)} \\ &\quad + \sup_{|t-s| \leq \delta, |x-y| \leq \varepsilon} |f^n(t - s, x - y, \lambda) - f^n(t, x, \lambda)| \\ &\leq L_K(n) + \mathcal{W}_K^n(\delta, \varepsilon). \end{aligned} \quad (2.41)$$

Thus, multiplying (2.34) by an arbitrary fixed nonnegative $\psi \in \mathcal{D}((2\varepsilon, \infty) \times \mathbb{R}^N)$, integrating the result over $\mathbb{R}_+ \times \mathbb{R}^N$ and using integration by parts, we find that

$$\begin{aligned} &- \int_{\mathbb{R}_+ \times \mathbb{R}^N} \left(\left\langle \omega_2^\varepsilon * \langle \eta(x, \lambda) - \eta(x, \mu) |, \nu_{(t,x)}^\delta(\lambda) \rangle_t, \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \right\rangle \right. \\ &\quad \left. + \left\langle \omega_2^\varepsilon * \langle \eta(x, \lambda) - \eta(x, \mu) |, \sigma_{(t,x)}^\delta(\mu) \rangle_t, \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \right\rangle \right) \psi dx dt \\ &\quad + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \left\langle Q(\lambda, \mu), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \otimes \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \right\rangle \cdot \nabla \psi dx dt \\ &\geq - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E'(\lambda - \mu)(f^n(t, x, \lambda) - f^n(t, x, \mu)), \nu_{(t,x)}^{\delta,\varepsilon}(\lambda) \otimes \sigma_{(t,x)}^{\delta,\varepsilon}(\mu) \rangle \psi dx dt \\ &\quad - 2 \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\mathcal{W}_K^n(\delta, \varepsilon) + L_K(n)) \psi dx dt \end{aligned} \quad (2.42)$$

First, we let $\varepsilon \rightarrow 0_+$. Then let $\Omega_\psi := \text{supp } \psi$. From (2.22) it follows that there exists a compact set K such that for $(t, x) \in \Omega_\psi$ we have $\text{supp } \nu_{(t,x)}^\delta \subset K$ and then also $\text{supp } \partial_t \nu_{(t,x)}^\delta \subset K$. The same holds for $\sigma_{(t,x)}^\delta$.

Since θ is bounded by some function independent of x , then there exists a function h_3 , again independent of x , such that for all $x \in \mathbb{R}^N$ and all $v \in \mathbb{R}$

$$|\eta(x, v)| \leq h_3(v),$$

which provides that $\eta \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \mathcal{C}(K))$, where $(t, x) \mapsto \eta(t, x, \cdot)$, hence also $\eta \in L^1(\Omega_\psi; \mathcal{C}(K))$. Consequently $E(\eta(x, \lambda) - \eta(x, \mu)) \in L^1(\Omega_\psi; \mathcal{C}(K))$. Thus we can extract a subsequence, that we do not relabel, such that

$$\begin{aligned} \omega_2^\varepsilon * \langle E(\eta(\cdot, \lambda) - \eta(\cdot, \mu)), \partial_t \nu^\delta \rangle &\rightarrow \langle E(\eta(\cdot, \lambda) - \eta(\cdot, \mu)), \partial_t \nu^\delta \rangle && \text{strongly in } L^1(\Omega_\psi; \mathcal{C}(K)), \\ \omega_2^\varepsilon * \langle E(\eta(\cdot, \lambda) - \eta(\cdot, \mu)), \partial_t \sigma^\delta \rangle &\rightarrow \langle E(\eta(\cdot, \lambda) - \eta(\cdot, \mu)), \partial_t \sigma^\delta \rangle && \text{strongly in } L^1(\Omega_\psi; \mathcal{C}(K)), \\ \sigma^{\delta, \varepsilon} &\rightharpoonup^* \sigma^\delta && \text{weakly}^* \text{ in } L_w^\infty(\Omega_\psi; \mathcal{M}(K)), \\ \nu^{\delta, \varepsilon} &\rightharpoonup^* \nu^\delta && \text{weakly}^* \text{ in } L_w^\infty(\Omega_\psi; \mathcal{M}(K)), \end{aligned}$$

as $\varepsilon \rightarrow 0$. Using these convergence results, we observe from (2.42) that

$$\begin{aligned} & - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}^\delta(\lambda) \rangle_t, \sigma_{(t,x)}^\delta(\mu) \rangle \psi \, dx \, dt \\ & - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \langle E(\eta(x, \lambda) - \eta(x, \mu)), \sigma_{(t,x)}^\delta(\mu) \rangle_t, \nu_{(t,x)}^\delta(\lambda) \rangle \psi \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle Q(\lambda, \mu), \nu_{(t,x)}^\delta(\lambda) \otimes \sigma_{(t,x)}^\delta(\mu) \rangle \cdot \nabla \psi \, dx \, dt \\ & \geq - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E'(\lambda - \mu)(f^n(t, x, \lambda) - f^n(t, x, \mu)), \nu_{(t,x)}^\delta(\lambda) \otimes \sigma_{(t,x)}^\delta(\mu) \rangle \psi \, dx \, dt \\ & - 2 \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\mathcal{W}_K^n(\delta, 0) + L_K(n)) \psi \, dx \, dt. \end{aligned} \tag{2.43}$$

Similarly to (2.27) it is not difficult to observe that

$$\begin{aligned} \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}^\delta \otimes \sigma_{(t,x)}^\delta \rangle_t &= \langle \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}^\delta \rangle, \sigma_{(t,x)}^\delta \rangle_t \\ &= \langle \omega^\delta * \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)} \rangle, \sigma_{(t,x)}^\delta \rangle_t \\ &= \langle (\omega^\delta * \langle \zeta, \nu_{(t,x)} \rangle)_t, \sigma_{(t,x)}^\delta \rangle + \langle (\omega^\delta * \langle \zeta, \sigma_{(t,x)} \rangle)_t, \nu_{(t,x)}^\delta \rangle. \end{aligned} \tag{2.44}$$

Thus, using (2.43), (2.44) and integrating by parts with respect to t , we find that

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}^\delta(\lambda) \otimes \sigma_{(t,x)}^\delta(\mu) \rangle \psi_t \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle Q(\lambda, \mu), \nu_{(t,x)}^\delta(\lambda) \otimes \sigma_{(t,x)}^\delta(\mu) \rangle \cdot \nabla \psi \, dx \, dt \\ & \geq - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E'(\lambda - \mu)(f^n(t, x, \lambda) - f^n(t, x, \mu)), \nu_{(t,x)}^\delta(\lambda) \otimes \sigma_{(t,x)}^\delta(\mu) \rangle \\ & - 2 \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\mathcal{W}_K^n(\delta, 0) + L_K(n)) \psi \, dx \, dt. \end{aligned}$$

Letting $\delta \rightarrow 0_+$ we conclude by the argument of weak* convergence of measures ν^δ and σ^δ to ν and σ , respectively and $\mathcal{W}_K^n(\delta, 0) \rightarrow 0$.

$$\begin{aligned}
& \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E(\eta(x, \lambda) - \eta(x, \mu)), \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \psi_t \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle Q(\lambda, \mu), \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \cdot \nabla \psi \, dx \, dt \\
& \geq - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle E'(\lambda - \mu)(f^n(t, x, \lambda) - f^n(t, x, \mu)), \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \psi \, dx \, dt \\
& - 2 \int_{\mathbb{R}_+ \times \mathbb{R}^N} L_K(n) \psi \, dx \, dt.
\end{aligned}$$

In the final step we let $n \rightarrow \infty$ and since $f^n \rightarrow f$ in $L^1(\Omega_\psi; \mathcal{C}(K))$ we obtain (2.26).

2.2 Comparison and contraction principles for entropy weak solutions

In the next lemma we included contraction and comparison principle for entropy weak solutions. In order not to involve the method of doubling the variables for weak solutions we use as much as possible the results obtained for measure-valued solutions. Here we consider the solutions v_1 and v_2 corresponding to the problems with different right-hand side. The purpose is to work later with approximated problems, where the source term shall be perturbed with a strictly monotone term and for the sake of constructing monotone families of approximated sequence we shall be interested in different parameters.

Lemma 2.2 *Assume that v_1, v_2 are two entropy weak solutions to (1.1) with a right-hand side f_1 and f_2 respectively. Then*

1. *for all nonnegative $\psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^N)$ it holds*

$$\begin{aligned}
& \int_{\mathbb{R}_+ \times \mathbb{R}^N} |\eta(x, v_1) - \eta(x, v_2)| \psi_t(t, x) \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(v_1 - v_2) (A(v_1) - A(v_2)) \cdot \nabla \psi(t, x) \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(v_1 - v_2) (f_1(t, x, v_1) - f_2(t, x, v_2)) \psi(t, x) \, dx \, dt \\
& \geq - \int_{\{(t,x): v_1=v_2\}} |f_1(t, x, v_1) - f_2(t, x, v_2)| \psi(t, x) \, dx \, dt
\end{aligned} \tag{2.45}$$

2. for all nonnegative $\psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^N)$ it holds

$$\begin{aligned}
& \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\eta(x, v_1) - \eta(x, v_2))^+ \psi_t(t, x) \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \chi_{\{v_1 > v_2\}} (A(v_1) - A(v_2)) \cdot \nabla \psi(t, x) \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \chi_{\{v_1 > v_2\}} (f_1(t, x, v_1) - f_2(t, x, v_2)) \psi(t, x) \, dx \, dt \\
& \geq - \int_{\{(t, x): v_1 = v_2\}} (f_1(t, x, v_1) - f_2(t, x, v_2))^+ \psi(t, x) \, dx \, dt
\end{aligned} \tag{2.46}$$

Proof: If v_1, v_2 are entropy weak solutions, then the Dirac masses $\delta_{v_1(t, x)}$ and $\delta_{v_2(t, x)}$ are corresponding entropy measure-valued solutions. Repeating step by step the argumentation from previous lemma we arrive at

$$\begin{aligned}
& - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \left(\left\langle \omega_2^\varepsilon * \langle |\eta(x, \lambda) - \eta(x, \mu)|, \delta_{v_1(t, x)}^\delta(\lambda) \rangle_t, \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \right\rangle \right. \\
& \quad \left. + \left\langle \omega_2^\varepsilon * \langle |\eta(x, \lambda) - \eta(x, \mu)|, \delta_{v_2(t, x)}^\delta(\mu) \rangle_t, \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \right\rangle \right) \psi \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \left\langle \operatorname{sgn}(\lambda - \mu) (A(\lambda) - A(\mu)), \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \otimes \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \right\rangle \cdot \nabla \psi \, dx \, dt \\
& \geq - \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(\lambda - \mu) (f_1^n(t, x, \lambda) - f_2^n(t, x, \mu)), \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \otimes \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle \psi \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\mathcal{K}_{\varepsilon, \delta, n}^\lambda + \mathcal{K}_{\varepsilon, \delta, n}^\mu) \psi \, dx \, dt
\end{aligned} \tag{2.47}$$

where

$$\begin{aligned}
\mathcal{K}_{\varepsilon, \delta, n}^\lambda & := \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\lambda - \mu) (f_1(t, x, \lambda) - f_1^n(t, x, \lambda)), \delta_{v_1(t, x)}^\delta(\lambda) \rangle, \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \rangle \\
& + \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\lambda - \mu) f_1^n(t, x, \lambda), \delta_{v_1(t, x)}^\delta(\lambda) \rangle, \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \rangle \\
& - \langle \langle E'(\lambda - \mu) f_1^n(t, x, \lambda), \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle, \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \rangle
\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}
\mathcal{K}_{\varepsilon, \delta, n}^\mu & := \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\mu - \lambda) (f_2(t, x, \mu) - f_2^n(t, x, \mu)), \delta_{v_2(t, x)}^\delta(\mu) \rangle, \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle \\
& + \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle E'(\mu - \lambda) f_2^n(t, x, \mu), \delta_{v_2(t, x)}^\delta(\mu) \rangle, \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle \\
& - \langle \langle E'(\mu - \lambda) f_2^n(t, x, \mu), \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \rangle, \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle
\end{aligned} \tag{2.49}$$

In what follows we shall only concentrate on the first integral on the right hand side of (2.47). The error estimates follow the same lines as (2.41).

Since the function $\text{sgn}(\lambda - \mu)(f_1^n(t, x, \lambda) - f_2^n(t, x, \mu))$ may fail to be continuous for $\lambda = \mu$, hence we shall discuss separately the integrals

$$I_1^{\delta, \varepsilon} := \int_{\{(t, x): v_1 \neq v_2\}} \langle \text{sgn}(\lambda - \mu)(f_1^n(t, x, \lambda) - f_2^n(t, x, \mu)), \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \otimes \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle \psi \, dx \, dt \quad (2.50)$$

and

$$I_2^{\delta, \varepsilon} := \int_{\{(t, x): v_1 = v_2\}} \langle \text{sgn}(\lambda - \mu)(f_1^n(t, x, \lambda) - f_2^n(t, x, \mu)), \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \otimes \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle \psi \, dx \, dt. \quad (2.51)$$

We let $\varepsilon \rightarrow 0_+$ and $\delta \rightarrow 0_+$. Then

$$\begin{aligned} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_1^{\delta, \varepsilon} &= \int_{\{(t, x): v_1 \neq v_2\}} \langle \text{sgn}(\lambda - \mu)(f_1^n(t, x, \lambda) - f_2^n(t, x, \mu)), \delta_{v_2(t, x)}(\mu) \otimes \delta_{v_1(t, x)}(\lambda) \rangle \psi \, dx \, dt \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}^N} \text{sgn}(v_1 - v_2)(f_1^n(t, x, v_1) - f_2^n(t, x, v_2)) \psi \, dx \, dt. \end{aligned} \quad (2.52)$$

where the last equality holds since $\text{sgn} 0 = 0$. The second integral can be estimated as follows

$$\begin{aligned} |I_2^{\delta, \varepsilon}| &\leq \int_{\{(t, x): v_1 = v_2\}} |\langle \text{sgn}(\lambda - \mu)(f_1^n(t, x, \lambda) - f_2^n(t, x, \mu)), \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \otimes \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle| \psi \, dx \, dt \\ &\leq \int_{\{(t, x): v_1 = v_2\}} \langle |f_1^n(t, x, \lambda) - f_2^n(t, x, \mu)|, \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \otimes \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle \psi \, dx \, dt \end{aligned} \quad (2.53)$$

and therefore

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_2^{\delta, \varepsilon} \leq \int_{\{(t, x): v_1 = v_2\}} |f_1^n(t, x, v_1) - f_2^n(t, x, v_2)| \psi \, dx \, dt. \quad (2.54)$$

which completes the proof of point 1.

To prove the second part of the theorem, again we shall argue on the level of measure-valued solutions. Now we will use the entropy inequalities both for convex and concave entropies. First observe that for all $\mu \in \mathbb{R}$

$$\begin{aligned} \omega_2^\varepsilon * \langle (\eta(x, \lambda) - \eta(x, \mu))^+, \delta_{v_1(t, x)}^\delta(\lambda) \rangle_t + \text{div} \langle \chi_{\{\lambda > \mu\}}(A(\lambda) - A(\mu)), \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle \\ \leq (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle \chi_{\{\lambda > \mu\}} f_1(t, x, \lambda), \delta_{v_1(t, x)}(\lambda) \rangle \end{aligned} \quad (2.55)$$

and all $\lambda \in \mathbb{R}$

$$\begin{aligned} -\omega_2^\varepsilon * \langle (\eta(x, \mu) - \eta(x, \lambda))^- , \delta_{v_2(t, x)}^\delta(\mu) \rangle_t + \text{div} \langle \chi_{\{\mu < \lambda\}}(A(\mu) - A(\lambda)), \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \rangle \\ \geq (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle \chi_{\{\mu < \lambda\}} f_2(t, x, \mu), \delta_{v_2(t, x)}(\mu) \rangle. \end{aligned} \quad (2.56)$$

Hence multiplying (2.56) by -1 and adding it to (2.55) we obtain

$$\begin{aligned}
& \langle \omega_2^\varepsilon * \langle (\eta(x, \lambda) - \eta(x, \mu))^+, \delta_{v_1(t, x)}^\delta(\lambda) \rangle_t, \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \rangle \\
& + \langle \omega_2^\varepsilon * \langle (\eta(x, \lambda) - \eta(x, \mu))^+, \delta_{v_2(t, x)}^\delta(\mu) \rangle_t, \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle \\
& + \operatorname{div} \langle \chi_{\{\lambda > \mu\}}(A(\lambda) - A(\mu)), \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \otimes \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \rangle \\
& \leq \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle \chi_{\{\lambda > \mu\}} f_1(t, x, \lambda), \delta_{v_1(t, x)}^\delta(\lambda) \rangle, \delta_{v_2(t, x)}^{\delta, \varepsilon}(\mu) \rangle \\
& - \langle (\omega_1^\delta \cdot \omega_2^\varepsilon) * \langle \chi_{\{\lambda > \mu\}} f_2(t, x, \mu), \delta_{v_2(t, x)}^\delta(\mu) \rangle, \delta_{v_1(t, x)}^{\delta, \varepsilon}(\lambda) \rangle
\end{aligned} \tag{2.57}$$

We repeat the same arguments as in the previous part of the proof.

3 Existence of entropy weak solutions

Proof of Theorem 1.3. The proof starts with the existence of entropy measure-valued solution, then we shall show that it is unique and is in fact an entropy weak solution.

We construct the approximate problem. Let now A^j be a sequence of smooth functions such that for every compact set $K \subset \mathbb{R}$

$$A^j \rightarrow A \quad \text{strongly in } \mathcal{C}(K; \mathbb{R}^N). \tag{3.58}$$

Let θ^* be a minimal selection of the graph of θ . We approximate θ in two steps. First we shall construct the Yosida approximation of θ with a parameter \sqrt{j} and then mollify this $\theta_{\sqrt{j}}$ with respect to x and u . Therefore let us define

$$J_{\frac{1}{\sqrt{j}}} = (\operatorname{id} + \frac{1}{\sqrt{j}} \theta)^{-1} \tag{3.59}$$

and

$$\theta_{\frac{1}{\sqrt{j}}} = \sqrt{j} (\operatorname{id} - J_{\frac{1}{\sqrt{j}}}). \tag{3.60}$$

Then

$$\theta^{(j)}(x, u) := \int_{\mathbb{R}^N \times \mathbb{R}} \omega^{\frac{1}{j}}(x - y, u - z) \theta_{\frac{1}{\sqrt{j}}}(y, z) dy dz, \tag{3.61}$$

where $\omega^{\frac{1}{j}}$ is the standard mollification kernel of radius $\frac{1}{j}$. To provide that the approximation vanishes at zero define

$$\theta^j(x, u) := \theta^{(j)}(x, u) - \theta^{(j)}(x, 0). \tag{3.62}$$

Observe that with such a choice of parameters we get

$$\theta^{(j)}(\cdot, 0) \rightarrow 0 \quad \text{a.e. in } \mathbb{R}^N \tag{3.63}$$

and we denote by $\eta^j(x, z)$ the inverse function to $\theta^j(x, u)$, i.e., $\eta^j(x, \theta^j(x, u)) = u$. Moreover, let

$$f^{(j)}(t, x, u) := \int_{\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}} \omega^{\frac{1}{j}}(t - s, x - y, u - z) f^{(j)}(s, y, z) ds dy dz \tag{3.64}$$

and define

$$f^j(t, x, u) := f^{(j)}(t, x, u) - f^{(j)}(t, x, 0). \quad (3.65)$$

Moreover, we will add a strictly dissipative perturbation term defined as follows

$$\varphi_{\ell, m}(r) := \frac{1}{\ell} \arctan(r^-) - \frac{1}{m} \arctan(r^+).$$

Hence the approximate problem has a form

$$u_t^j + \operatorname{div} A^j(\theta^j(x, u^j)) = f^j(t, x, u^j) + \varphi_{\ell, m}(\theta^j(x, u^j)) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^N, \quad (3.66)$$

$$u^j(0, \cdot) = u_0 \quad \text{on } \mathbb{R}^N. \quad (3.67)$$

We will divide the proof into three steps. In the first step we shall concentrate on existence of measure-valued solutions (namely we will pass with $j \rightarrow \infty$), in the second step we will show that the measure-valued solution is indeed an entropy weak solution to the problem with a strictly dissipative perturbation and in the final third step we will pass to the limit with $\ell, m \rightarrow \infty$ and conclude existence of entropy weak solution to the original problem.

Step 1. Existence of solutions is provided by the classical theory of Kruřkov, cf. [15]. Since condition (1.3) holds, with the standard estimates one gets that for any j θ^j , $\frac{\partial}{\partial x_i} \theta^j = \theta_{\frac{1}{\sqrt{j}}} * \frac{\partial}{\partial x_i} \omega^{\frac{1}{j}}$ is bounded in $\mathbb{R}^N \times [-M, M]$ for all $M > 0$ and the assumptions of [15] are satisfied. By Lemma A.1 we can define

$$v^j(t, x) := \theta^j(x, u^j(t, x)) \quad (3.68)$$

which satisfies for all nonnegative $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^N)$ the entropy inequality

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} |\eta^j(x, v^j(t, x)) - \eta^j(x, k)| \psi_t(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(v^j(t, x) - k) (A^j(v^j(t, x)) - A^j(k)) \cdot \nabla \psi(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(v^j(t, x) - k) (f^j(t, x, \eta^j(x, v^j)) + \varphi_{\ell, m}(v^j)) \psi \, dx \, dt \\ & + \int_{\mathbb{R}^N} |u_0(x) - \eta^j(x, k)| \psi(0, x) \, dx \geq 0. \end{aligned} \quad (3.69)$$

Since u^j is bounded in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$, then by (1.4) the sequence v^j is also bounded. From the entropy inequality (3.69) we want to pass to the following entropy inequalities

$$\begin{aligned}
& \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\eta^j(x, v^j(t, x)) - \eta^j(x, k))^+ \psi_t(t, x) \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \chi_{\{v^j(t, x) > k\}} (A^j(v^j(t, x)) - A^j(k)) \cdot \nabla \psi(t, x) \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \chi_{\{v^j(t, x) > k\}} (f^j(t, x, \eta^j(x, v^j)) + \varphi_{\ell, m}(v^j)) \psi \, dx \, dt \\
& + \int_{\mathbb{R}^N} (u_0(x) - \eta^j(x, k))^+ \psi(0, x) \, dx \geq 0
\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\eta^j(x, v^j(t, x)) - \eta^j(x, k))^- \psi_t(t, x) \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \chi_{\{v^j(t, x) < k\}} (A^j(v^j(t, x)) - A^j(k)) \cdot \nabla \psi(t, x) \, dx \, dt \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \chi_{\{v^j(t, x) < k\}} (f^j(t, x, \eta^j(x, v^j)) + \varphi_{\ell, m}(v^j)) \psi \, dx \, dt \\
& + \int_{\mathbb{R}^N} (u_0(x) - \eta^j(x, k))^- \psi(0, x) \, dx \leq 0
\end{aligned} \tag{3.71}$$

satisfied for all nonnegative $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^N)$. For this purpose we first choose in (3.69) $k = \|v^j\|_{L^\infty}$ and $k = -\|v^j\|_{L^\infty}$, which allows to conclude that the problem

$$\begin{aligned}
\eta^j(x, v^j)_t + \operatorname{div} A^j(v^j) &= f^j(t, x, \eta(x, v^j)) + \varphi_{\ell, m}(v^j), \\
v^j(0, x) &= \theta^j(x, u_0)
\end{aligned} \tag{3.72}$$

is satisfied in a distributional sense. Obviously the following problem

$$\eta^j(x, k)_t + \operatorname{div} A^j(k) = 0, \tag{3.73}$$

with initial condition k is satisfied in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^N)$. Hence a linear combination of (3.69), (3.72) and (3.73) allows to conclude (3.70) and (3.71).

We want to pass to the limit with $j \rightarrow \infty$ in (3.70) (and (3.71) respectively, which we do not present in detail since it is easily concluded from the first part). Obviously, there exists a subsequence (labelled the same) and $v \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ such that

$$v^j \xrightarrow{*} v \quad \text{in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^N). \tag{3.74}$$

Moreover there exists a Young measure $\nu_{(t, x)}$ associated to the subsequence v^j . In the remaining part of this step of the proof we will show that ν is an entropy measure-valued solution in the sense of Definition 2.1. To pass to the limit in (3.70) ((3.71) follows

analogously) with the first terms on the right-hand side we first make an observation on θ^j , namely due to (3.63) for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$ where θ is single-valued and continuous with respect to u

$$\theta^j \rightarrow \theta^* \quad (3.75)$$

a.e. in \mathbb{R}^N . Hence there exists $M \subset \mathbb{R}^N$ such that $|M| = 0$ such that

$$\theta^j(x, \cdot) \rightarrow \theta^*(x, \cdot)$$

for all $x \in \mathbb{R}^N \setminus M$. The strict monotonicity of θ^j with respect to the last variable allows to conclude with help of Proposition A.3 for a.a. $x \in \mathbb{R}^N$ the locally uniform convergence of $\eta^j(x, \cdot)$. Define

$$\zeta^j(x, s) := (\eta^j(x, s) - \eta^j(x, k))^+ \quad (3.76)$$

and

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\mathbb{R}_+ \times \mathbb{R}^N \setminus M} \zeta^j(x, v^j) dx dt &= \lim_{j \rightarrow \infty} \langle \zeta^j, v^j \rangle \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}^N} \int_{\mathbb{R}} (\eta(x, \lambda) - \eta(x, k))^+ d\nu_{(t,x)}(\lambda) dx dt \end{aligned} \quad (3.77)$$

where the duality pairing is understood between the spaces $L^1(\mathbb{R}^d; \mathcal{C}((-R, R); \mathbb{R}))$ and $L_w^\infty(\mathbb{R}^d; \mathcal{M}([-R, R]))$. The limit passage in the second term of (3.70) and (3.71) follows the same lines as in [5].

We direct our attention to the limit passage in the term containing f^j . The main problem is the appearance of a discontinuous function

$$\lambda \mapsto \chi_{\{\lambda > \mu\}}(f^j(t, x, \eta^j(x, \lambda)) + \varphi_{\ell, m}(\lambda)).$$

For this purpose we shall construct a family of functions which allow to estimate the discontinuous term. We will call it $\chi_{\{\lambda > \mu\}}^\gamma$ and define as follows: for $\mu \geq 0$

$$\chi_{\{\lambda > \mu\}}^{\gamma, +}(\lambda) := \begin{cases} \chi_{\{\lambda > \mu\}} & \text{for } \lambda < \mu, \lambda \geq \mu + \gamma, \\ \text{affine} & \text{for } \mu \leq \lambda < \mu + \gamma. \end{cases} \quad (3.78)$$

For $\mu < 0$

$$\chi_{\{\lambda > \mu\}}^{\gamma, -}(\lambda) := \begin{cases} \chi_{\{\lambda > \mu\}} & \text{for } \lambda < \mu - \gamma, \lambda \geq \mu, \\ \text{affine} & \text{for } \mu - \gamma \leq \lambda < \mu. \end{cases} \quad (3.79)$$

Note that since $f + \varphi_{\ell, m}$ are dissipative, then the above definition of $\chi_{\{\lambda > \mu\}}^\gamma$ provides that

$$\chi_{\{\lambda > \mu\}}^\gamma(f(t, x, \lambda) + \varphi_{\ell, m}(\lambda)) \geq \chi_{\{\lambda > \mu\}}(f(t, x, \lambda) + \varphi_{\ell, m}(\lambda)) \quad (3.80)$$

for any $\lambda \in \mathbb{R}$, therefore inequality (3.70) with $\chi_{\{\lambda > \mu\}}^\gamma$ instead of $\chi_{\{\lambda > \mu\}}$ in the third term on the left-hand side holds. The convergence of convolutions and the dissipativity/monotonicity of f , f^j and η, η^j provide that $f^j(t, x, \eta^j(x, \lambda))$ converges a.e with respect to t and x and uniformly with respect to λ on a bounded interval $[-R, R]$ to the function f , see Proposition A.2, namely

$$f^j(\cdot, \cdot, \eta^j) \rightarrow f(\cdot, \cdot, \eta) \quad \text{strongly in } L_{loc}^1(\mathbb{R}_+ \times \mathbb{R}^N; \mathcal{C}([-R, R])). \quad (3.81)$$

We obtain that

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\mathbb{R}_+ \times \mathbb{R}^N} \chi_{\{v^j(t,x) > k\}}^\gamma (f^j(t,x, \eta^j(x, v^j)) + \varphi_{\ell,m}(v^j)) \psi \, dx \, dt \\ = \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda > \mu\}}^\gamma (f(t,x, \lambda) + \varphi_{\ell,m}(\lambda)), \nu_{(t,x)}(\lambda) \rangle \psi \, dx \, dt. \end{aligned} \quad (3.82)$$

Then we pass with $\gamma \rightarrow 0_+$. The limit passage is obvious for those μ that $\nu_{(t,x)}(\{\mu\}) \stackrel{\text{a.e.}}{=} 0$ on $\text{supp } \psi$. Let again $\Omega_\psi := \text{supp } \psi$, where ψ has compact support in $\mathbb{R}_+ \times \mathbb{R}^N$ hence $|\Omega_\psi| < \infty$. We shall now concentrate on showing that the set

$$I := \{\mu \in \mathbb{R} : |\{(t,x) \in \Omega_\psi : \nu_{(t,x)}(\{\mu\}) > 0\}| > 0\}$$

is at most countable. Indeed, assume the opposite. If $\nu_{(t,x)}(\{\mu\}) > 0$ on some subset of $\mathbb{R}_+ \times \mathbb{R}^N$ of positive measure, then $\int_{\Omega_\psi} \nu_{(t,x)}(\{\mu\}) \, dx \, dt > 0$, but also

$$\sum_{\mu \in I} \int_{\Omega_\psi} \nu_{(t,x)}(\{\mu\}) \, dx \, dt \leq \int_{\Omega_\psi} \int_{\mathbb{R}} 1 \, d\nu_{(t,x)} \, dx \, dt.$$

Since the set I is not countable, then the series diverges, but we know that the Young measure ν is a probability measure, therefore the right-hand side equals to $|\Omega_\psi|$ and we obtain a contradiction. Consequently the set $\mathbb{R} \setminus I$ is a dense set in \mathbb{R} . We conclude that for all $\mu \in \mathbb{R} \setminus I$ and all nonnegative $\psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^N)$

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle (\eta(x, \lambda) - \eta(x, \mu))^+, \nu_{(t,x)}(\lambda) \rangle \psi_t(t, x) \, dx \, dt \\ + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda > \mu\}} (A(\lambda) - A(\mu)), \nu_{(t,x)}(\lambda) \rangle \cdot \nabla \psi(t, x) \, dx \, dt \\ + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda > \mu\}} (f(t, x, \lambda) + \varphi_{\ell,m}(\lambda)), \nu_{(t,x)}(\lambda) \rangle \psi \, dx \, dt \geq 0. \end{aligned} \quad (3.83)$$

To claim that the above inequality holds for all $\mu \in \mathbb{R}$ observe that the function

$$\mu \mapsto \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle (\eta(x, \lambda) - \eta(x, \mu))^+, \nu_{(t,x)}(\lambda) \rangle \psi_t(t, x) \, dx \, dt$$

as well as

$$\mu \mapsto \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda > \mu\}} (A(\lambda) - A(\mu)), \nu_{(t,x)}(\lambda) \rangle \cdot \nabla \psi(t, x) \, dx \, dt$$

are continuous w.r.t. μ . Observe now the function

$$\mu \mapsto \int_{\mathbb{R}_+ \times \mathbb{R}^N} \langle \chi_{\{\lambda > \mu\}} (f(t, x, \lambda) + \varphi_{\ell,m}(\lambda)), \nu_{(t,x)}(\lambda) \rangle \psi \, dx \, dt, \quad (3.84)$$

which is not continuous w.r.t. μ , but one can notice it is decreasing/increasing depending on the sign of μ . For this purpose let us split the integral as follows

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} \int_{\mathbb{R}_+} \chi_{\{\lambda > \mu\}} (f(t, x, \lambda) + \varphi_{\ell, m}(\lambda)) d\nu_{(t, x)}(\lambda) \psi dx dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \int_{\mathbb{R}_-} \chi_{\{\lambda > \mu\}} (f(t, x, \lambda) + \varphi_{\ell, m}(\lambda)) d\nu_{(t, x)}(\lambda) \psi dx dt. \end{aligned} \quad (3.85)$$

Depending on the sign of μ , always the terms $\chi_{\{\lambda > \mu\}}$ in one of the above integrals will be constant. In the second integral, because of the dissipativity of f and $\varphi_{\ell, m}$, we know the sign of the integrand, which allows to claim that the function (3.84) is monotone w.r.t. μ . Therefore if we take $\mu \in I, \mu > 0$, then one can find a sequence μ^n such that

$$\begin{aligned} \lim_{\mu^n \rightarrow \mu^-} \int_{\mathbb{R}_+ \times \mathbb{R}^N} \int_{\mathbb{R}} \chi_{\{\lambda > \mu^n\}} f(t, x, \lambda) d\nu_{(t, x)}(\lambda) & \geq \int_{\mathbb{R}_+ \times \mathbb{R}^N} \int_{\mathbb{R}} \chi_{\{\lambda > \mu\}} f(t, x, \lambda) d\nu_{(t, x)}(\lambda) \\ & \geq \lim_{\mu^n \rightarrow \mu^+} \int_{\mathbb{R}_+ \times \mathbb{R}^N} \int_{\mathbb{R}} \chi_{\{\lambda > \mu^n\}} f(t, x, \lambda) d\nu_{(t, x)}(\lambda). \end{aligned} \quad (3.86)$$

For $\mu \in I, \mu < 0$ the inequalities hold in an opposite direction. Analogously one can show that (2.24) holds.

Step 2. Let now ν, σ be two entropy measure-valued solutions. By Lemma 2.1 we obtain that (2.26) holds with $f(t, x, \lambda) + \varphi_{\ell, m}$ instead of $f(t, x, \lambda)$. Let $0 < \varepsilon < t_0 < T < \infty$ be arbitrary. We define an affine $\psi_{\varepsilon, t_0}^1$ as follows

$$\psi_{\varepsilon, t_0}^1(t) := \begin{cases} 0 & t \in [0, t_0 - \varepsilon) \cup [T, \infty), \\ \frac{t - t_0 + \varepsilon}{\varepsilon} & t \in (t_0 - \varepsilon, t_0), \\ \frac{T - t}{T - t_0} & t \in (t_0, T). \end{cases}$$

Let $\psi_2^n \in \mathcal{D}(\mathbb{R}^d)$ be arbitrary such that $\|\psi_2^n\|_\infty \leq 1$. Then we set $\psi(t, x) := \psi_{\varepsilon, t_0}^1(t) \psi_2^n(x)$ in (2.26) (it is a possible test function since we can mollify ψ_1 and then pass to the limit). Hence, using for simplicity the notation $f_{\ell, m}(t, x, \lambda) := f(t, x, \lambda) + \varphi_{\ell, m}(\lambda)$ and $Q(\lambda, \mu) = \text{sgn}(\lambda - \mu)(A(\lambda) - A(\mu))$

$$\begin{aligned} & \frac{1}{T - t_0} \int_{t_0}^T \int_{\mathbb{R}^N} \langle |\eta(x, \lambda) - \eta(x, \mu)|, \nu_{(t, x)}(\lambda) \otimes \sigma_{(t, x)}(\mu) \rangle \psi_2^n(x) dx dt \\ & \leq \frac{1}{\varepsilon} \int_{t_0 - \varepsilon}^{t_0} \int_{\mathbb{R}^N} \langle |\eta(x, \lambda) - \eta(x, \mu)|, \nu_{(t, x)}(\lambda) \otimes \sigma_{(t, x)}(\mu) \rangle \psi_2^n(x) dx dt \\ & + \int_{t_0 - \varepsilon}^T \int_{\mathbb{R}^N} \langle Q(\lambda, \mu), \nu_{(t, x)}(\lambda) \otimes \sigma_{(t, x)}(\mu) \rangle \cdot \nabla \psi_2^n(x) \psi_1(t) dx dt \\ & + \int_{t_0 - \varepsilon}^T \int_{\mathbb{R}^N} \langle \text{sgn}(\lambda - \mu) (f_{\ell, m}(t, x, \lambda) - f_{\ell, m}(t, x, \mu)), \nu_{(t, x)}(\lambda) \otimes \sigma_{(t, x)}(\mu) \rangle \cdot \psi_2^n(x) \psi_{\varepsilon, t_0}^1 dt dx \end{aligned}$$

Our goal is to let $\varepsilon \rightarrow 0_+$, and next $t_0 \rightarrow 0_+$. Because of the initial condition (see (2.25)) and continuity of the solution in appropriate topology the first term on the right-hand side above will vanish. Considerations concerning the left-hand side and the second term on the right-hand side follow the same lines as in [5]. There is no problem to pass to the limit in the term with $f_{\ell,m}$. For arbitrary $\psi_2^n \in \mathcal{D}(\mathbb{R}^N)$ such that $\|\psi_2^n\|_\infty \leq 1$ and any $T > 0$ at the limit we find

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \langle |\eta(x, \lambda) - \eta(x, \mu)|, \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \psi_2^n(x) \, dx \, dt \\ & \leq T \int_0^T \int_{\mathbb{R}^N} \langle |Q(\lambda, \mu)|, \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle |\nabla \psi_2^n(x)| \, dx \, dt \\ & + T \int_0^T \int_{\mathbb{R}^N} \langle \operatorname{sgn}(\lambda - \mu) (f_{\ell,m}(t, x, \lambda) - f_{\ell,m}(t, x, \mu)) \rangle, \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \cdot \psi_1(t) \psi_2^n(x) \, dx \, dt. \end{aligned} \quad (3.87)$$

Where $\psi_1(t) = 1 - \frac{t}{T}$ for $t \in [0, T]$. Note that the term on the left-hand side is non-negative. Because of the growth conditions that were assumed on A , we conclude that $\langle |Q(\lambda, \mu)|, \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \in L^1(0, T; L^p(\mathbb{R}^N))$. Finally, we define a monotone sequence $\psi_2^n \nearrow 1$ of smooth nonnegative compactly supported functions as $\psi_2^n(x) := 1$ in $B(0, n)$, $\psi_2^n(x) := 0$ for $x \in \mathbb{R}^N \setminus B(0, 2n)$ such that $|\nabla \psi_2^n| \leq \frac{c}{n}$. For handling the flux term one immediately observes that

$$\int_{\mathbb{R}^N} |\nabla \psi_2^n|^q \, dx \leq C \quad \text{for all } q \geq N,$$

and

$$|\nabla \psi_2^n| \rightharpoonup^* 0 \text{ weakly}^* \text{ in } L^\infty(0, T; L^q(\mathbb{R}^N)) \quad \text{for all } q \geq N, \quad (3.88)$$

which is enough that this term vanishes. With the monotone convergence theorem we conclude that

$$0 \leq \int_0^T \int_{\mathbb{R}^N} \langle \operatorname{sgn}(\lambda - \mu) (f_{\ell,m}(t, x, \lambda) - f_{\ell,m}(t, x, \mu)) \rangle, \nu_{(t,x)}(\lambda) \otimes \sigma_{(t,x)}(\mu) \rangle \psi_1(t) \, dx \, dt \leq 0. \quad (3.89)$$

Because of the strict dissipativity of the function $f_{\ell,m}$ the last inequality is strict except of the diagonal. Since the left-hand side is nonnegative, there exists a function

$$v \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N) \quad (3.90)$$

such that

$$\nu_{t,x} = \sigma_{t,x} = \delta_{v(t,x)} \quad \text{for a.a. } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N. \quad (3.91)$$

Hence we conclude that for each ℓ, m there exists an entropy weak solution.

Step 3. In the final step we will pass with $\ell, m \rightarrow \infty$. Let then $v_{\ell,m}$ and $v_{\ell,m'}$ be entropy weak solutions to the problems with a right-hand side $f + \varphi_{\ell,m}$ and $f + \varphi_{\ell,m'}$

respectively with $m' > m$. We will now use inequality (2.46) for the solutions $v_{\ell,m}$ and $v_{\ell,m'}$. We proceed with choosing a test function in the same way and limit passage with $\varepsilon \rightarrow 0_+$ and next $t_0 \rightarrow 0_+$ as in the previous step. Hence

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}_+ \times \mathbb{R}^N} (\eta(x, v_{\ell,m}) - \eta(x, v_{\ell,m'}))^+ dx dt \\
&\leq T \int_{\mathbb{R}_+ \times \mathbb{R}^N} \chi_{\{v_{\ell,m} > v_{\ell,m'}\}} (f_{\ell,m}(t, x, v_{\ell,m}) - f_{\ell,m'}(t, x, v_{\ell,m'})) \psi_1(t) dx dt \\
&\quad + T \int_{\{(t,x): v_{\ell,m} = v_{\ell,m'}\}} (f_{\ell,m}(t, x, v_{\ell,m}) - f_{\ell,m'}(t, x, v_{\ell,m'}))^+ \psi_1(t) dx dt.
\end{aligned} \tag{3.92}$$

The second term on the right-hand side can be neglected since

$$\begin{aligned}
&\chi_{\{v_{\ell,m} = v_{\ell,m'}\}} (f_{\ell,m}(t, x, v_{\ell,m}) - f_{\ell,m'}(t, x, v_{\ell,m'}))^+ \\
&= \chi_{\{v_{\ell,m} = v_{\ell,m'}\}} \left(f_{\ell,m'}(t, x, v_{\ell,m}) + \left(\frac{1}{m'} - \frac{1}{m} \right) \arctan(v_{\ell,m}^+) - f_{\ell,m'}(t, x, v_{\ell,m'}) \right)^+ \\
&= \chi_{\{v_{\ell,m} = v_{\ell,m'}\}} \left(\left(\frac{1}{m'} - \frac{1}{m} \right) \arctan(v_{\ell,m}^+) \right)^+ = 0.
\end{aligned} \tag{3.93}$$

Observe now the first term on the right-hand side

$$\begin{aligned}
&\chi_{\{v_{\ell,m} > v_{\ell,m'}\}} (f_{\ell,m}(t, x, v_{\ell,m}) - f_{\ell,m'}(t, x, v_{\ell,m'})) \\
&= \chi_{\{v_{\ell,m} > v_{\ell,m'}\}} \left(f_{\ell,m'}(t, x, v_{\ell,m}) + \left(\frac{1}{m'} - \frac{1}{m} \right) \arctan(v_{\ell,m}^+) - f_{\ell,m'}(t, x, v_{\ell,m'}) \right) \\
&\leq \chi_{\{v_{\ell,m} > v_{\ell,m'}\}} (f_{\ell,m'}(t, x, v_{\ell,m}) - f_{\ell,m'}(t, x, v_{\ell,m'})) \leq 0
\end{aligned} \tag{3.94}$$

where the last inequality holds since $m' > m$, $\arctan(v_{\ell,m}^+) \geq 0$ and the function $f_{\ell,m'}$ is dissipative. Therefore, since $\psi_1(t)$ is nonnegative, then

$$\chi_{\{v_{\ell,m} > v_{\ell,m'}\}} (f_{\ell,m'}(t, x, v_{\ell,m}) - f_{\ell,m'}(t, x, v_{\ell,m'})) = 0 \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}^N. \tag{3.95}$$

Strict dissipativity of $f_{\ell,m'}$ allows to conclude that

$$v_{\ell,m} \leq v_{\ell,m'} \tag{3.96}$$

In the same manner, choosing $\ell' > \ell$ one shows that

$$v_{\ell',m} \leq v_{\ell,m}, \tag{3.97}$$

where $v_{\ell',m}, v_{\ell,m}$ are entropy weak solutions to the problems with a right-hand side $f_{\ell',m}$ and $f_{\ell,m}$. We will pass to the limit with $m \rightarrow \infty$ and then with $\ell \rightarrow \infty$. The monotonicity provides that for each ℓ there exists a limit v_ℓ such that

$$v_{\ell,m} \rightarrow v_\ell \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}^N. \tag{3.98}$$

Hence, if we denote $v_{\ell'}$ a limit of a sequence $v_{\ell',m}$, then from (3.97) we conclude that

$$v_{\ell} \leq v_{\ell'} \quad (3.99)$$

for $\ell' > \ell$. Hence as $\ell \rightarrow \infty$

$$v_{\ell} \rightarrow v \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}^N. \quad (3.100)$$

3.1 Uniqueness of entropy weak solutions

Using the local comparison principle of Lemma 2.2 we obtain uniqueness of the entropy weak solution. Let us assume that u_1 and u_2 are entropy weak solutions to (1.1) in the sense of Definition 1.1. Then we take $\psi = \psi_{\varepsilon,t_0}^1(t)\psi_2^n(x)$ as a test function in (2.45), where ψ_{ε,t_0}^1 and ψ_2^n are defined as in the proof of Theorem 3 and we repeat the argumentation of this proof, Step 2 to pass to the limit with $\varepsilon \rightarrow 0^+$ and $t_0 \downarrow 0$ using the initial condition. Finally we choose $\psi_2^n(x)$ to be a smooth approximation of $\chi_{\mathbb{R}^N}$ and pass to the limit with $n \rightarrow \infty$ repeating the arguments of the proof of Theorem 3, Step 2.

A Equivalent notions of entropy solutions

In this section we concentrate on relations between different notions of entropy weak solutions for the flux function Φ in a form $\Phi(x, u) = A(\theta(x, u))$ with A, θ satisfying (H1)–(H3) with an additional condition that A is sufficiently regular in both variables. These relations play an important role on the level of approximations, namely after passing from discontinuous flux to sufficiently smooth one. We formulate the lemma collecting the relations between different notions of solutions.

Lemma A.1 *Let Φ, f satisfy the assumptions (H1)–(H4) and assume that $A \in C^1(\mathbb{R})$, θ is continuous in u and continuously differentiable in x . Assume that $u \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}^N)$ is given and define*

$$v(t, x) := \theta(x, u(t, x)), \quad (A.101)$$

$$(A.102)$$

Then the following statements are equivalent.

(N1) *For all $k \in \mathbb{R}$ and all nonnegative $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^N)$ there holds*

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} |u(t, x) - k| \psi_t(t, x) - \operatorname{sgn}(u(t, x) - k) \operatorname{div} \Phi(x, k) \psi(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(u(x, t) - k) (\Phi(x, u(x, t)) - \Phi(x, k)) \cdot \nabla \psi(x, t) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(u(t, x) - k) f(t, x, u) \psi \, dx \, dt + \int_{\mathbb{R}^N} |u_0(x) - k| \psi(0, x) \, dx \geq 0. \end{aligned} \quad (A.103)$$

(N2) For all $k \in \mathbb{R}$ and all nonnegative $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^N)$ there holds

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}^N} |\eta(x, v(t, x)) - \eta(x, k)| \psi_t(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(v(t, x) - k) (A(v(t, x)) - A(k)) \cdot \nabla \psi(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}_+ \times \mathbb{R}^N} \operatorname{sgn}(v(t, x) - k) f(t, x, \eta(x, v)) \psi \, dx \, dt + \int_{\mathbb{R}^N} |u_0(x) - \eta(x, k)| \psi(0, x) \, dx \geq 0. \end{aligned} \quad (\text{A.104})$$

Proof: To show **(N1)** \Rightarrow **(N2)** consider the equation

$$(u_i)_t + \operatorname{div} \Phi(x, u_i) = f_i(t, x, u_i), \quad i = 1, 2.$$

For any two entropy weak solutions u_1, u_2 the so-called Kato inequality holds

$$\begin{aligned} & |u_1 - u_2|_t + \operatorname{div} (\operatorname{sgn}(u_1 - u_2) (\Phi(x, u_1) - \Phi(x, u_2))) \\ & \leq \operatorname{sgn}(u_1 - u_2) (f_1 - f_2) + |f_1 - f_2| \chi_{\{u_1 = u_2\}} \end{aligned} \quad (\text{A.105})$$

in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^N)$, cf. Ref. [10]. Choosing in (A.105) $u_1 = \eta(x, v_1)$ and $u_2 = \eta(x, k)$ with $f_1 = f$, $f_2 \equiv 0$. Note that the set of $k \in \mathbb{R}$ such that $|\{(t, x) : u_1(t, x) = \eta(x, k)\}| > 0$ is at most countable and hence it allows to pass from (A.105) to **(N2)**.

For showing the opposite direction let us consider the problem with $f_i : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying Lipschitz condition with respect to the last variable.

$$\eta(x, v_i)_t + \operatorname{div} A(v_i) = f_i(t, x, \eta(x, v_i)), \quad i = 1, 2. \quad (\text{A.106})$$

For any entropy weak solutions v_1, v_2 in the sense of **(N2)** it holds, cf. (2.46)

$$\begin{aligned} & |\eta(x, v_1) - \eta(x, v_2)|_t + \operatorname{div} (\operatorname{sgn}(v_1 - v_2) (A(v_1) - A(v_2))) \\ & \leq \operatorname{sgn}(v_1 - v_2) (f_1 - f_2) + |f_1 - f_2| \chi_{\{v_1 = v_2\}} \end{aligned} \quad (\text{A.107})$$

in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^N)$. For passing from (A.107) to **(N1)** we choose again $u_1 = \eta(x, v_1)$ and now $v_2 = \theta(x, k)$ with $f_1 = f$ and $f_2 = \operatorname{div} A(v_2) = \operatorname{div} \Phi(x, k)$. Note again that the set of $k \in \mathbb{R}$ such that $|\{(t, x) : v_2(t, x) = \theta(x, k)\}| > 0$ is at most countable and we pass to **(N1)** what completes the proof.

Proposition A.2 Let $[a, b] \subset \mathbb{R}$ and let f be continuous, f, f_n be monotone functions such that $f_n \rightarrow f$ pointwisely. Then $f_n \rightarrow f$ uniformly on $[a, b]$.

The above fact is an elementary exercise. For the proof see e.g. [1].

Proposition A.3 Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $\operatorname{Im}(f_n) = \mathbb{R}$, f_n be strictly monotone functions. Let f be a maximal monotone mapping with $\operatorname{Im}(f) = \mathbb{R}$ and let the inverse mapping f^{-1} be continuous and $f_n \rightarrow f$ a.e.. Then the inverse functions converge locally uniformly to the inverse of the limit, namely $(f_n)^{-1} \rightarrow f^{-1}$ uniformly on every compact subset of \mathbb{R} .

Proof: We provide the proof by contradiction. Assume that f_n converges a.e. to f and that $(f_n)^{-1}$ does not converge pointwisely to f^{-1} . Hence there exist $y, \varepsilon > 0$ and a subsequence $(f_{n_k})^{-1}$ such that

$$(f_{n_k})^{-1}(y) \notin [f^{-1}(y) - \bar{\varepsilon}, f^{-1}(y) + \bar{\varepsilon}]. \quad (\text{A.108})$$

for every $0 < \bar{\varepsilon} < \varepsilon$.

Case 1: $(f_{n_k})^{-1}(y) > f^{-1}(y) + \bar{\varepsilon}$. Let $z = f^{-1}(y)$, hence $f(z) \ni y$ and there exists a selection f^* such that $y = f^*(z)$. Define $\bar{y}_{n_k} := f_{n_k}^{-1}(f^*(z))$. By (A.108) we have the estimate

$$\bar{y}_{n_k} > z + \bar{\varepsilon}.$$

Using the strict monotonicity of f (hence obviously also f^*), monotonicity of f_{n_k} and the definition of \bar{y}_{n_k} we conclude an existence of δ such that for every $\bar{\varepsilon} \in (\frac{\varepsilon}{2}, \varepsilon)$

$$0 < \delta \leq f^*(z + \bar{\varepsilon}) - f^*(z) = f^*(z + \bar{\varepsilon}) - f_{n_k}(\bar{y}_{n_k}) \leq f(z + \bar{\varepsilon}) - f_{n_k}(z + \bar{\varepsilon}). \quad (\text{A.109})$$

Since the number of points where f is not continuous is countable, it is always possible to choose such $\bar{\varepsilon}$ that $z + \bar{\varepsilon}$ is the point where f is continuous (single valued). Hence for such $\bar{\varepsilon}$ (A.109) contradicts the convergence of f_n .

Case 2: $(f_{n_k})^{-1}(y) < f^{-1}(y) - \bar{\varepsilon}$. Let again $z = f^{-1}(y)$, and $y = f^*(z)$. Define $\bar{y}_{n_k} := f_{n_k}^{-1}(f^*(z))$ and observe that

$$\bar{y}_{n_k} < z - \bar{\varepsilon}.$$

Again we conclude an existence of δ such that for every $\bar{\varepsilon} \in (\frac{\varepsilon}{2}, \varepsilon)$

$$0 < \delta \leq f^*(z) - f^*(z - \bar{\varepsilon}) = f_{n_k}(\bar{y}_{n_k}) - f^*(z - \bar{\varepsilon}) \leq f_{n_k}(z - \bar{\varepsilon}) - f(z - \bar{\varepsilon}). \quad (\text{A.110})$$

and we conclude in the same way as in the previous case. Hence $(f_n)^{-1}$ converges pointwisely to f^{-1} . The uniform convergence of $(f_n)^{-1}$ can be concluded by Proposition A.2.

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